When mathematics helps physics: Calculation of the integral of Kholodenko and Silagadze 🄵

D. Beck ✉ iD

Check for updates

View Online   Export Citation

Articles You May Be Interested In

Inability of the hypernetted chain integral equation to exhibit a spinodal line
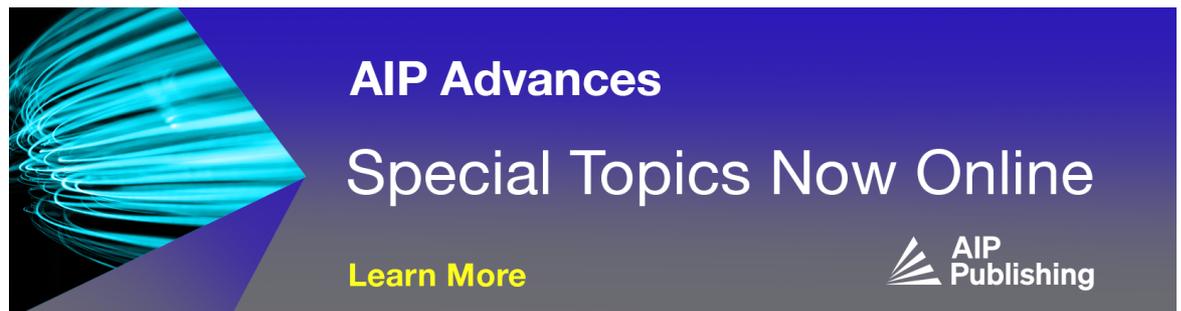
J. Chem. Phys. (May 1993)

Bosonic field propagators on algebraic curves

J. Math. Phys. (September 2000)

Thermodynamics and structure of a two-dimensional electrolyte by integral equation theory

J. Chem. Phys. (May 2014)

When mathematics helps physics: Calculation of the integral of Kholodenko and Silagadze Ⓕ

View Online   Export Citation   CrossMark

D. Beck[a] ⅮD

AFFILIATIONS

Charles University, Faculty of Mathematics and Physics, Prague, Czechia

[a] Author to whom correspondence should be addressed: beckd@karlin.mff.cuni.cz. **URL:** https://www.karlin.mff.cuni.cz/~beckd/

ABSTRACT

In this paper, we show a physics-free derivation of a Landau–Zener type integral introduced by Kholodenko and Silagadze.

I. INTRODUCTION

In 2012, Kholodenko and Silagadze published an enigmatic paper[1] in which they showed for $n = 1, 2, 3, \ldots$ that

$$I_n = \int_{-\infty}^{\infty} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_{2n-1}} \cos\left(x_1^2 - x_2^2\right) \cos\left(x_3^2 - x_4^2\right) \cdots$$
$$\times \cos\left(x_{2n-1}^2 - x_{2n}^2\right) \, dx_{2n} \cdots dx_2 dx_1 = \frac{2}{n!}\left(\frac{\pi}{4}\right)^n. \tag{1}$$

Their approach involved an intricate use of the concepts from quantum physics, namely, spinorization and the Hopf map, using which they were able to show that the integral is equivalent to the *Landau–Zener formula*. The Landau–Zener formula expresses the exact probability amplitude of a certain quantum system with a dynamical potential and its standard derivation uses either the contour integration (Landau's approach) or asymptotics of a certain differential equation (Zener's approach); see the work of Piquer i Méndez[2] for a clear exposition of both approaches. Alternatively, see the work of Rojo[3] for a matrix exponential solution or that of Glasbrenner and Schleich[4] for a derivation using Markov property.

We were aware of the result (1) as early as 2015, in which we solved an integral related to I_4 also introduced by Silagadze.[5] Our derivation can be seen here.[6] There is also an alternative derivation of the I_4 related integral due to de Reyna[7] using clever asymptotics of a certain sine-type power series.

However, back then, we lacked a proper mathematical background to tackle the general problem of I_n using only simple and purely mathematical concepts as we did in the case of I_4. Now, exactly ten years later, we are returning to the problem with a new

toolkit: contour integration. Unlike the original Landau derivation, we will focus only on the integral without mentioning its connection to quantum physics. Despite its ubiquitousness and broad applications in quantum physics, simple derivations of Landau–Zener formula are rare and many of them do not use clear assumptions. Therefore, the purpose of our paper is to present a physics-free derivation, mainly to make the result accessible to a broader mathematical community.

II. PRELIMINARIES
A. Auxiliary functions

In our paper, we define four auxiliary functions of a real parameter t, which turn out to be relevant,

$$U(t) = \int_0^\infty \frac{\sqrt{x}}{1+x^2} \exp\left(\frac{t}{4}\arctan x\right) dx,$$
$$V(t) = \int_0^\infty \frac{1/\sqrt{x}}{1+x^2} \exp\left(\frac{t}{4}\arctan x\right) dx,$$
$$P(t) = \int_0^1 \frac{\sqrt{y}-1}{1-y^2} \exp\left(\frac{it}{4}\operatorname{argtanh} y\right) dy, \tag{2}$$
$$Q(t) = \int_0^1 \frac{\frac{1}{\sqrt{y}}-1}{1-y^2} \exp\left(\frac{it}{4}\operatorname{argtanh} y\right) dy.$$

Those functions can be expressed in an exact form using Appell and generalized hypergeometric functions. The exact expressions are, however, not necessary for our derivation of I_n. Note that those

functions are not independent. By substituting $x \to 1/x$, we obtain an obvious relation between U and V,

$$U(t) = e^{\pi t/8} \, V(-t). \tag{3}$$

Moreover, the complex-valued functions P, Q can be expressed as linear combinations of real-valued-only functions U and V, respectively. In order to find those relations, we need to study a certain contour integral.

B. Contour integral

1. Definitions

Consider the following contour integral with parameters $A, B \in \mathbb{C}$, and $t \in \mathbb{R}$:

$$\oint_C \left[\frac{\sqrt{z} - \sqrt{i}}{1 + z^2} A + \frac{\frac{1}{\sqrt{z}} - \frac{1}{\sqrt{i}}}{1 + z^2} B \right] e^{\frac{t}{4} \arctan z} \, dz, \tag{4}$$

where we define $\arctan z = \frac{1}{2i} \ln \left(\frac{1+iz}{1-iz} \right)$. Both $\sqrt{z}$ and $\ln z$ functions are assumed to have branch cuts at the negative real axis (the so called *principal branch*), that is, $\arg z \in (-\pi, \pi]$. Before proceeding further, let us clarify all the functions appearing in the contour integral. Readers familiar with complex functions may skip the following paragraphs. Writing $z = re^{i\varphi}$ with $r > 0$ and $\varphi \in (-\pi, \pi]$, the complex square root function is defined by

$$\sqrt{re^{i\varphi}} = \sqrt{r} e^{i\varphi/2}, \tag{5}$$

so $\sqrt{i}$ in the formula above is given by

$$\sqrt{i} = e^{i\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i. \tag{6}$$

Similarly, we have for the complex logarithm,

$$\ln \left(re^{i\varphi} \right) = \ln r + i\varphi. \tag{7}$$

Those definitions ensure both $\sqrt{z}$ and $\ln z$ are analytic on $\mathbb{C} \backslash (-\infty, 0]$.

2. Real axis

Note that we recover the ordinary arctan function on reals. To see this, let $z = x$, $x \in \mathbb{R}$, then $\arg z \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$ and thus

$$1 \pm ix = \sqrt{1 + x^2} e^{\pm i \arctan x}. \tag{8}$$

Substituting into the complex definition of arctan,

$$\frac{1}{2i} \ln e^{2i \arctan x} = \frac{1}{2i} \left(2i \arctan x \right) = \arctan x \tag{9}$$

since $2 \arctan x \in (-\pi, \pi)$.

3. Branch cuts

To locate the branch cuts of the complex function $\arctan z$, we need to find which points on the z-plane are mapped onto the negative real axis in $w = \frac{1+iz}{1-iz}$ as $\arctan z = \frac{1}{2i} \ln w$. Solving

$$\frac{1 + iz}{1 - iz} = -y, \quad y > 0, \tag{10}$$

we get the location of the branch cut in $z \in \mathbb{C}$ as

$$z = i \frac{1+y}{1-y} \in (-\infty i, -i) \cup (i, i\infty), \tag{11}$$

where (z_1, z_2) denotes a complex interval from z_1 to z_2 (a line segment between any $z_1, z_2 \in \mathbb{C}$).

4. Imaginary axis

Let $z = iy, y > 0$, so $z = ye^{\pi i/2}$ and thus

$$\sqrt{z} = \sqrt{iy} = \sqrt{y} e^{\pi i/4} = \sqrt{i} \sqrt{y}. \tag{12}$$

The situation is different for the arctan function since it is no longer analytic on the whole imaginary axis. Let $z = iy, y \in (0, 1)$ (analytic part), then

$$\arctan z = \frac{1}{2i} \ln \left(\frac{1-y}{1+y} \right) = i \, \mathrm{argtanh} \, y, \tag{13}$$

where $\mathrm{argtanh} \, y = \frac{1}{2} \ln \frac{1+y}{1-y}$ is the usual inverse hyperbolic tangent function. As we have derived, the branch cut of $\arctan z$ function is located at the imaginary axis. Let us study how the function behaves in its neighbourhood. Let $z = iy + \varepsilon, y > 1$ with $\varepsilon > 0$ small positive (so z is taken to lie to the right from the branch cut). By Taylor expansion,

$$\left. \frac{1+iz}{1-iz} \right|_{iy+\varepsilon} = \left. \frac{1+iz}{1-iz} \right|_{iy} + \varepsilon \left. \left(\frac{1+iz}{1-iz} \right)' \right|_{iy} + O(\varepsilon^2)$$

$$= -\frac{y-1}{y+1} + \frac{2i\varepsilon}{(y-1)^2} = \frac{y-1}{y+1} e^{\pi i}, \tag{14}$$

which gives

$$\arctan z = \frac{1}{2i} \left(\ln \frac{y-1}{y+1} + \pi i \right) = \frac{\pi}{2} + i \, \mathrm{argtanh} \, \frac{1}{y}. \tag{15}$$

5. Asymptotics

Let $|z| \to \infty$, expanding $\frac{1+iz}{1-iz}$, we get

$$\frac{1+iz}{1-iz} = -1 + \frac{2i}{z} + O\left(\frac{1}{z^2} \right) = e^{\pi i} \left(1 - \frac{2i}{z} \right) + O\left(\frac{1}{z^2} \right). \tag{16}$$

If $\Re z \to \infty$ only (that is $z \to \infty e^{i\varphi}$ with $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$), we get $\arg (1 - 2i/z) \in (-\pi, 0)$ and thus

$$\ln \left(\frac{1+iz}{1-iz} \right) = \ln \left(e^{\pi i} \right) + \ln \left(1 - \frac{2i}{z} \right) = \pi i - \frac{2i}{z} + O\left(\frac{1}{z^2} \right), \tag{17}$$

from which, dividing by $2i$,

$$\arctan z = \frac{\pi}{2} - \frac{1}{z} + O\left(\frac{1}{z^2} \right), \quad \Re z \to \infty. \tag{18}$$

6. Pole expansion

Let us examine the behavior of $\arctan z$ near its singularity $z = i$ in the first quadrant in $\mathbb{C}$. Let $z = i - i\varepsilon e^{i\varphi}, \varepsilon > 0$ and $\varphi \in (0, \pi)$. Then,

$$\frac{1+iz}{1-iz} = \frac{1-1+\varepsilon e^{i\varphi}}{1+1-\varepsilon e^{i\varphi}} = \frac{\varepsilon}{2} e^{i\varphi} + O(\varepsilon^2), \tag{19}$$

from which

$$\arctan z = \frac{1}{2i}\ln\frac{1+iz}{1-iz} = \frac{1}{2i}\ln\frac{\varepsilon}{2} + \frac{\varphi}{2} + O(\varepsilon^2). \qquad (20)$$

Next, the square brackets part of $f(z)$ has a finite limit as $z \to i$. By L'Hospitals rule,

$$\lim_{z\to i}\frac{\sqrt{z}-\sqrt{i}}{1+z^2} = \left.\frac{\frac{1}{2\sqrt{z}}}{2z}\right|_{z=i} = \frac{1}{4i\sqrt{i}}, \qquad (21)$$

similarly for $(\frac{1}{\sqrt{z}} - \frac{1}{\sqrt{i}})/(1+z^2)$ as $z \to i$.

C. Parameterization

We are now ready to examine integral (4). It is convenient to denote

$$f(z) = \left[\frac{\sqrt{z}-\sqrt{i}}{1+z^2}A + \frac{\frac{1}{\sqrt{z}}-\frac{1}{\sqrt{i}}}{1+z^2}B\right]e^{\frac{t}{4}\arctan z}. \qquad (22)$$

Let us consider the counterclockwise contour C consisting of integration curves $C_k, k = 1, \ldots, 5$,

$$C = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5, \qquad (23)$$

as shown in Fig. 1.

Inside C, function f is analytic, so by the Cauchy integral formula,

$$\oint_C f(z)\,\mathrm{d}z = \sum_{k=1}^{5}\int_{C_k} f(z)\,\mathrm{d}z = 0. \qquad (24)$$

Let us now parameterize the integrals on the individual integration curves; we denote $J_k = \int_{C_k} f(z)\,\mathrm{d}z$ and $\ominus C_k$ as negatively oriented C_k.

- $C_1 : z = x, \ x \in (0, R), \ R \to \infty, \ \mathrm{d}z = \mathrm{d}x,$

$$J_1 = \int_0^R\left[\frac{\sqrt{x}-\sqrt{i}}{1+x^2}A + \frac{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{i}}}{1+x^2}B\right]e^{\frac{t}{4}\arctan x}\,\mathrm{d}x. \qquad (25)$$

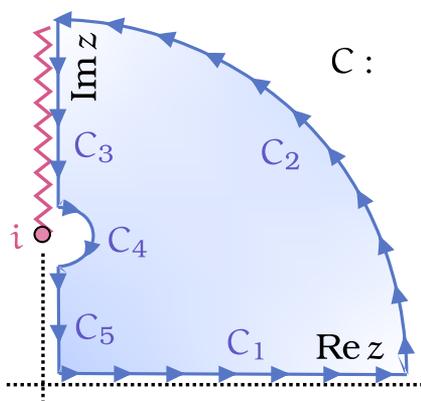**FIG. 1.** Contour C.

As $R \to \infty$, we may express the limit using the auxiliary functions defined earlier,

$$J_1 \to AU(t) + BV(t) - \sqrt{i}(A-iB)\int_0^\infty\frac{e^{\frac{t}{4}\arctan x}}{1+x^2}\,\mathrm{d}x. \qquad (26)$$

The integral on the right is trivial, we get

$$J_1 \to AU(t) + BV(t) - \frac{4\sqrt{i}}{t}(A-iB)\left(e^{\frac{\pi t}{8}}-1\right). \qquad (27)$$

- $C_2 : z = Re^{i\varphi}, R \to \infty, \varphi \in (0, \frac{\pi}{2}), \mathrm{d}z = iRe^{i\varphi}\mathrm{d}\varphi,$

$$|J_2| \le R\int_0^{\frac{\pi}{2}}\left[\frac{\sqrt{R}+1}{R^2-1}|A| + \frac{\frac{1}{\sqrt{R}}+1}{R^2-1}|B|\right]e^{\frac{\pi|t|}{8}}\,\mathrm{d}\varphi, \qquad (28)$$

from which $J_2 \to 0$ as $R \to \infty$.

- $\ominus C_3 : z = iy + \varepsilon, \ y \in (1+\varepsilon, R), \ \mathrm{d}z = i\mathrm{d}y,$

$$J_3 = -i\sqrt{i}\int_{1-\varepsilon}^R \exp\left(\frac{\pi t}{8} + \frac{it}{4}\operatorname{argtanh} y\right)$$
$$\times\left[\frac{\sqrt{y}-1}{1-y^2}A - i\frac{\frac{1}{\sqrt{y}}-1}{1-y^2}B\right]\mathrm{d}y. \qquad (29)$$

Substituting $y \to 1/y$ and letting $R \to \infty$ and $\varepsilon \to 0^+$, we get, using auxiliary functions,

$$J_3 \to i\sqrt{i}e^{\frac{\pi t}{8}}\left(AQ(t) - iBP(t)\right). \qquad (30)$$

- $\ominus C_4 : z = i - i\varepsilon e^{i\varphi}, \varphi \in (0, \pi), \varepsilon \to 0^+, \mathrm{d}z = \varepsilon e^{i\varphi}\mathrm{d}\varphi,$

$$|J_4| \le \varepsilon\int_0^\pi\left[\frac{|A|}{4} + \frac{|B|}{4} + O(\varepsilon)\right]e^{\frac{t}{8i}\ln\frac{\varepsilon}{2}+\frac{\varphi t}{8}}\,\mathrm{d}\varphi. \qquad (31)$$

Since $t \in \mathbb{R}$, J_4 vanishes as $\varepsilon \to 0^+$.

- $\ominus C_5 : z = iy, y \in (1, 1-\varepsilon), \varepsilon \to 0^+, \mathrm{d}z = i\mathrm{d}y,$

$$J_5 = -i\sqrt{i}\int_0^{1-\varepsilon}\left[\frac{\sqrt{y}-1}{1-y^2}A - i\frac{\frac{1}{\sqrt{y}}-1}{1-y^2}B\right]e^{\frac{it}{4}\operatorname{argtanh} y}\,\mathrm{d}y. \qquad (32)$$

As $\varepsilon \to 0^+$, we get, using auxiliary functions,

$$J_5 \to -i\sqrt{i}(AP(t) - iBQ(t)). \qquad (33)$$

D. Relations connecting the auxiliary functions

By the Cauchy integral formula, $J_1 + \cdots + J_5 = 0$. Taking the limit as $R \to \infty$ and $\varepsilon \to 0^+$, we get for any $A, B \in \mathbb{C}$,

$$AU(t) + BV(t) = \frac{4\sqrt{i}}{t}(A-iB)\left(e^{\frac{\pi t}{8}}-1\right)$$
$$+ i\sqrt{i}\left(AP(t) - iBQ(t) - e^{\frac{\pi t}{8}}AQ(t) + ie^{\frac{\pi t}{8}}BP(t)\right). \qquad (34)$$

Comparing the terms with the coefficient A and B, respectively, we get

$$U(t) = \frac{4\sqrt{i}}{t}\left(e^{\frac{\pi t}{8}} - 1\right) + i\sqrt{i}\left(P(t) - e^{\frac{\pi t}{8}}Q(t)\right),$$
$$V(t) = -\frac{4i\sqrt{i}}{t}\left(e^{\frac{\pi t}{8}} - 1\right) + \sqrt{i}\left(Q(t) - e^{\frac{\pi t}{8}}P(t)\right). \tag{35}$$

Solving for P and Q, we get, finally,

$$P(t) = -\frac{4i}{t} + \sqrt{i}\ \frac{U(t) + ie^{\frac{\pi t}{8}}V(t)}{e^{\frac{\pi t}{4}} - 1},$$
$$Q(t) = -\frac{4i}{t} + \sqrt{i}\ \frac{e^{\frac{\pi t}{8}}U(t) + iV(t)}{e^{\frac{\pi t}{4}} - 1}. \tag{36}$$

III. DERIVATION

A. Integral equation formulation

Relation (1) can be written recursively by introducing new functions τ_n as

$$\tau_n(x) = \int_{-\infty}^{x}\int_{-\infty}^{x_1}\cos\left(x_1^2 - x_2^2\right)\tau_{n-1}(x_2)\ dx_2 dx_1, \tag{37}$$

with initial condition $\tau_0(x) = 1$. We have

$$I_n = \tau_n(\infty), \quad n = 1, 2, 3, \ldots. \tag{38}$$

Let us introduce a new parameter $t \in \mathbb{R}$ using which we define a generating function T as

$$T(x, t) = \sum_{n=0}^{\infty}\tau_n(x)t^n. \tag{39}$$

Equation (37) then gets transformed into

$$T(x, t) = 1 + t\int_{-\infty}^{x}\int_{-\infty}^{x_1}\cos\left(x_1^2 - x_2^2\right)T(x_2, t)dx_2 dx_1, \tag{40}$$

which is an integral equation for $T(x, t)$ subject to the boundary condition,

$$T(-\infty, t) = 1. \tag{41}$$

The solution of (40) at $x \to \infty$ completely solves our problem of finding I_n since by expanding $T(\infty, t)$ in t when $t \to 0$, we get

$$T(\infty, t) = \sum_{n=0}^{\infty}\tau_n(\infty)t^n = 1 + \sum_{n=1}^{\infty}I_n t^n. \tag{42}$$

Note that rescaling by $\sqrt{s}$ with $s > 0$ and letting $s \to \infty$, we get the *stretching relations* of $T(x, t)$,

$$\lim_{s\to\infty}T\left(\frac{x}{\sqrt{s}}, t\right) = T(0, t),$$
$$\lim_{s\to 0^+}T\left(\frac{x}{\sqrt{s}}, t\right) = T(-\infty, t)\mathbf{1}_{x<0} + T(\infty, t)\mathbf{1}_{x>0}; \tag{43}$$

we will see their importance later.

B. Differential equation formulation

It is convenient to denote

$$\alpha(t) = \int_{-\infty}^{0}\cos\left(x^2\right)T(x, t)\ dt,$$
$$\beta(t) = \int_{-\infty}^{0}\sin\left(x^2\right)T(x, t)\ dt. \tag{44}$$

Differentiating the integral Eq. (40) with respect to x, we get

$$\frac{\partial T}{\partial x} = t\int_{-\infty}^{x}\cos\left(x^2 - y^2\right)T(y, x)\ dy. \tag{45}$$

At $x = 0$, we get $\frac{\partial T}{\partial x}(0, t) = t\alpha(t)$. Second differentiation with respect of x gives

$$\frac{\partial^2 T}{\partial x^2} = tT(x, t) - 2xt\int_{-\infty}^{x}\sin\left(x^2 - y^2\right)T(y, t)\ dy. \tag{46}$$

At $x = 0$, we get $\frac{\partial^2 T}{\partial x^2}(0, t) = tT(0, t)$. Yet, another differentiation yields

$$\frac{\partial^3 T}{\partial x^3} = t\frac{\partial T}{\partial x} - 4x^2\frac{\partial T}{\partial x} - 2t\int_{-\infty}^{x}\sin\left(x^2 - y^2\right)T(y, t)dy, \tag{47}$$

which gives $\frac{\partial^3 T}{\partial x^3}(0, t) = t^2\alpha(t) - 2t\beta(t)$ at $x = 0$. One final differentiation results in the following differential equation:

$$\frac{\partial^4 T}{\partial x^4} = (t - 4x^2)\frac{\partial^2 T}{\partial x^2} - 12x\frac{\partial T}{\partial x}. \tag{48}$$

This is an ordinary differential equation (ODE) with parameter t. Its solution at $x \to \infty$ completely solves our problem of finding I_n (expanding $T(\infty, t)$ in t). For completeness, the following is the complete list of *boundary conditions* on $T(x, t)$:

$$T(-\infty, t) = 1, \qquad \frac{\partial^2 T}{\partial x^2}(0, t) = tT(0, t),$$
$$\frac{\partial T}{\partial x}(0, t) = t\alpha(t), \qquad \frac{\partial^3 T}{\partial x^3}(0, t) = t^2\alpha(t) - 2t\beta(t).$$

Using the Mathematica `NDSolve` command, we can see numerically how the solution looks like for various parameters t (Fig. 2).
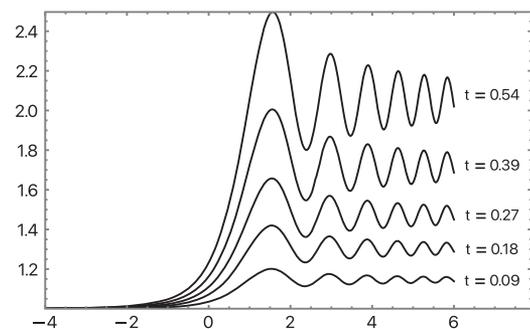
FIG. 2. Numerical solution of ODE for $T(x, t)$.

C. Even integral transform

In what follows, we establish a relation between $T(\infty, t)$ and $T(0, t)$. Let

$$e(s, t) = \int_{-\infty}^{\infty} T(x, t)\, e^{-sx^2}\, dx. \tag{49}$$

Since $T(x, t)$ is continuous, bounded, and having a finite limit $T(\infty, t)$, the integral on the right converges for any $s \in \mathbb{C}$ with $\Re s > 0$. Assuming s is a positive real number, we get by substitution $x = y/\sqrt{s}$,

$$e(s, t) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} T(\frac{y}{\sqrt{s}}, t) e^{-y^2}\, dy. \tag{50}$$

It is thus convenient to define $E(s, t) = e(s, t)\sqrt{s}$. Imposing the stretching relations (43),

$$E(\infty, t) = \int_{-\infty}^{\infty} T(0, t) e^{-y^2}\, dy = \sqrt{\pi}\, T(0, t), \tag{51}$$

$$E(0^+, t) = \int_{-\infty}^{\infty} (T(-\infty, t)\mathbf{1}_{y<0} + T(\infty, t)\mathbf{1}_{y>0}) e^{-y^2} dy = \frac{\sqrt{\pi}}{2} (T(\infty, t) + 1). \tag{52}$$

Differentiating $e(s, t)$ with respect to s and integrating by parts with respect to x yields

$$-e(s, t) - 2s\frac{\partial e}{\partial s} = \int_{-\infty}^{\infty} x\frac{\partial T}{\partial x} e^{-sx^2}\, dx \tag{53}$$

or in terms of $E(s, t)$,

$$-2\sqrt{s}\frac{\partial E}{\partial s} = \int_{-\infty}^{\infty} x\frac{\partial T}{\partial x} e^{-sx^2}\, dx. \tag{54}$$

Writing out the right-hand side using Eq. (45) and changing the order of integration,

$$-2\sqrt{s}\frac{\partial E}{\partial s} = t\int_{-\infty}^{\infty} T(y, t)\int_{y}^{\infty} \cos(x^2 - y^2)xe^{-sx^2} dx dy. \tag{55}$$

The inner integral is trivial and is equal to $\frac{s}{2(1+s^2)} e^{-sy^2}$. Hence, we get a differential equation for $E(s, t)$,

$$\frac{\partial E}{\partial s} = -\frac{tE(s, t)}{4(1+s^2)}. \tag{56}$$

The solution of this ODE with the appropriate boundary condition $E(\infty, t) = \sqrt{\pi}\, T(0, t)$ must be

$$E(s, t) = \sqrt{\pi}\, T(0, t) \exp\left(\frac{t}{4} \operatorname{arccot} s\right). \tag{57}$$

The remaining condition on $E(0^+, t)$ Eq. (52), therefore, yields as $s \to 0^+$,

$$T(\infty, t) = 2T(0, t) \exp\left(\frac{\pi t}{8}\right) - 1. \tag{58}$$

D. Truncated integral transform

In this section, we establish a relation between $T(-\infty, t) = 1$ and $T(0, t)$. Let

$$d(s, t) = \int_{-\infty}^{\infty} T(x, t)\, e^{-sx^2} \mathbf{1}_{x<0}\, dx. \tag{59}$$

That is, the integration domain is $(-\infty, 0)$. Again, the integral converges for any $s \in \mathbb{C}$ with $\Re s > 0$. Assuming s is a positive real number, we get by substitution $x = y/\sqrt{s}$,

$$d(s, t) = \frac{1}{\sqrt{s}} \int_{-\infty}^{0} T(\frac{y}{\sqrt{s}}, t) e^{-y^2}\, dy. \tag{60}$$

It is thus convenient to define analogously $D(s, t) = d(s, t)\sqrt{s}$. Imposing the stretching relations (43),

$$D(\infty, t) = \int_{-\infty}^{0} T(0, t) e^{-y^2}\, dy = \frac{\sqrt{\pi}}{2}\, T(0, t), \tag{61}$$

$$D(0^+, t) = \int_{-\infty}^{0} T(-\infty, t) e^{-y^2} dy = \frac{\sqrt{\pi}}{2}. \tag{62}$$

Differentiating $d(s, t)$ with respect to s and integrating by parts with respect to x yields

$$-d(s, t) - 2s\frac{\partial d}{\partial s} = \int_{-\infty}^{\infty} x\frac{\partial T}{\partial x} e^{-sx^2} \mathbf{1}_{x<0}\, dx \tag{63}$$

or in terms of $D(s, t)$,

$$-2\sqrt{s}\frac{\partial D}{\partial s} = \int_{-\infty}^{\infty} x\frac{\partial T}{\partial x} e^{-sx^2} \mathbf{1}_{x<0}\, dx. \tag{64}$$

Writing out the right-hand side using Eq. (45) and changing the order of integration,

$$-2\sqrt{s}\frac{\partial D}{\partial s} = t\int_{-\infty}^{0} T(y, t)\int_{y}^{0} \cos(x^2 - y^2)xe^{-sx^2} dx dy. \tag{65}$$

The inner integral is again trivial and is equal to

$$\frac{se^{-sy^2}}{2(1+s^2)} - \frac{s\cos(y^2) + \sin(y^2)}{2(1+s^2)}. \tag{66}$$

Hence, using the definition of $\alpha(t)$ and $\beta(t)$ Eq. (44), we get a differential equation for $D(s, t)$,

$$\frac{\partial D}{\partial s} = -\frac{tD(s, t)}{4(1+s^2)} + \frac{t}{4\sqrt{s}}\left(\frac{s\alpha(t)}{1+s^2} + \frac{\beta(t)}{1+s^2}\right). \tag{67}$$

The solution of this ODE with the boundary condition $D(0^+, t) = \sqrt{\pi}/2$ is

$$D(s, t) = \frac{\sqrt{\pi}}{2}\, e^{-\frac{s}{4}\arctan s}\left[1 + \frac{t}{2\sqrt{\pi}}\int_0^s \frac{1}{\sqrt{r}}\right.$$
$$\left. \times \left(\frac{r\alpha(t)}{1+r^2} + \frac{\beta(t)}{1+r^2}\right)e^{\frac{t}{4}\arctan r} dr\right]. \tag{68}$$

The remaining condition on $D(\infty, t)$ Eq. (61), therefore, yields plugging $s \to \infty$,

$$T(0,t)e^{\frac{\pi t}{8}} = 1 + \frac{t}{2\sqrt{\pi}}(\alpha(t)U(t) + \beta(t)V(t)), \tag{69}$$

where U and V are defined by Eq. (2).

E. Imaginary boundary conditions

The key to our problem is the realization that the solution we found for $D(s,t)$ is valid for all $s \in \mathbb{C}$ for which $\Re s > 0$. The functions $\alpha(t)$ and $\beta(t)$ are not chosen arbitrarily since they are defined using $T(x,t)$. We may also express them Eq. (44) in terms of $d(s,t)$ as

$$
\begin{aligned}
\alpha(t) \mp i\beta(t) &= \int_{-\infty}^{0} T(x,t)(\cos x^2 \mp i \sin x^2)\mathrm{d}x \\
&= \int_{-\infty}^{0} T(x,y)e^{\mp ix^2}\mathrm{d}x = d(\pm i, t).
\end{aligned} \tag{70}
$$

The value $d(\pm i, t)$ must be viewed as a limit of $d(s,t)$ for $s \to \pm i$ (since $\Re s > 0$). This gives us the final boundary condition. Considering only the case $s \to i$, we get in terms of $D(s,t) = d(s,t)\sqrt{s}$,

$$D(i,t) = \sqrt{i}\,(\alpha(t) - i\beta(t)). \tag{71}$$

On the other hand, $D(i,t)$ can be calculated directly from the solution we found for $D(s,t)$. Let $s = i\sigma, \sigma \in (0,1)$, then by Eq. (68),

$$
\begin{aligned}
D(i\sigma, t) = \frac{\sqrt{\pi}}{2}e^{-\frac{it}{4}\operatorname{argtanh}\sigma}\bigg[1 + \frac{i\sqrt{it}}{2\sqrt{\pi}} \\
\times \int_0^\sigma \left(\frac{\sqrt{y}}{1-y^2}\alpha(t) - \frac{\frac{1}{\sqrt{y}}}{1-y^2}i\beta(t)\right)e^{-\frac{it}{4}\operatorname{argtanh}\sigma}\,\mathrm{d}y\bigg].
\end{aligned} \tag{72}
$$

Writing

$$\frac{\sqrt{y}}{1-y^2} = \frac{\sqrt{y}-1}{1-y^2} + \frac{1}{1-y^2}, \qquad \frac{\frac{1}{\sqrt{y}}}{1-y^2} = \frac{\frac{1}{\sqrt{y}}-1}{1-y^2} + \frac{1}{1-y^2} \tag{73}$$

and integrating out the $1/(1-y^2)$ term, we get

$$
\begin{aligned}
D(i\sigma, t) = \sqrt{i}(\alpha(t) - i\beta(t)) + \frac{\sqrt{\pi}}{2}e^{-\frac{it}{4}\operatorname{argtanh}\sigma} \\
\times \bigg[1 - \frac{2\sqrt{i}}{\sqrt{\pi}}(\alpha(t) - i\beta(t)) + \frac{i\sqrt{i}\,t}{2\sqrt{\pi}} \\
\times \int_0^\sigma\left(\frac{\sqrt{y}-1}{1-y^2}\alpha(t) - \frac{\frac{1}{\sqrt{y}}-1}{1-y^2}i\beta(t)\right)e^{-\frac{it}{4}\operatorname{argtanh}\sigma}\mathrm{d}y\bigg].
\end{aligned} \tag{74}
$$

Note that the integral is now finite as $\sigma \to 1$. Crucially, as σ approaches 1, since the first part already yields the correct limit $\sqrt{i}(\alpha(t) - i\beta(t))$, the term in the square bracket must vanish, otherwise, the limit as a whole would not exist. Hence, we get the following condition:

$$1 = \frac{2\sqrt{i}}{\sqrt{\pi}}(\alpha(t) - i\beta(t)) - \frac{i\sqrt{i}t}{2\sqrt{\pi}}(\alpha(t)P(t) - i\beta(t)Q(t)), \tag{75}$$

where P and Q are defined by Eq. (2). Substituting for P and Q from Eq. (36), we get

$$
\begin{aligned}
\frac{2\sqrt{\pi}}{t}\left(e^{\frac{\pi t}{4}} - 1\right) &= \alpha(t)U(t) + \beta(t)V(t) \\
&\quad + ie^{\frac{\pi t}{8}}(\alpha(t)V(t) - \beta(t)U(t)).
\end{aligned} \tag{76}
$$

Comparing the real and imaginary parts, we obtain

$$
\begin{aligned}
\alpha(t)U(t) + \beta(t)V(t) &= \frac{2\sqrt{\pi}}{t}\left(e^{\frac{\pi t}{4}} - 1\right), \\
\alpha(t)V(t) - \beta(t)U(t) &= 0,
\end{aligned} \tag{77}
$$

which is a linear system of equations for $\alpha(t)$ and $\beta(t)$, with the solution given as

$$
\begin{aligned}
\alpha(t) &= \frac{2\sqrt{\pi}\left(e^{\frac{\pi t}{4}} - 1\right)U(t)}{t\left(U(t)^2 + V(t)^2\right)}, \\
\beta(t) &= \frac{2\sqrt{\pi}\left(e^{\frac{\pi t}{4}} - 1\right)V(t)}{t\left(U(t)^2 + V(t)^2\right)}.
\end{aligned} \tag{78}
$$

F. Final matching

Substituting $\alpha(t)$ and $\beta(t)$ into Eq. (69),

$$T(0,t) = \exp\left(\frac{\pi t}{8}\right), \tag{79}$$

from which, by Eq. (58),

$$T(\infty, t) = 2\exp\left(\frac{\pi t}{4}\right) - 1. \tag{80}$$

The series expansion of $T(\infty, t)$ Eq. (42) immediately yields

$$I_n = \frac{2}{n!}\left(\frac{\pi}{4}\right)^n. \tag{81}$$

IV. RELATED RESULT

A similar analysis also shows that

$$
\begin{aligned}
K_n = \int_{-\infty}^\infty \int_{x_1}^\infty \cdots \int_{x_{2n-1}}^\infty \cos\left(x_1^2 - x_2^2\right)\cos\left(x_3^2 - x_4^2\right)\cdots \\
\times \cos\left(x_{2n-1}^2 - x_{2n}^2\right)\mathrm{d}x_{2n}\cdots\mathrm{d}x_2\mathrm{d}x_1 = \frac{2}{n!}\left(\frac{\pi}{4}\right)^n.
\end{aligned} \tag{82}
$$

11 July 2025 03:32:46

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

D Beck: Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES

[1] A. L. Kholodenko and Z. K. Silagadze, "When physics helps mathematics: Calculation of the sophisticated multiple integral," Phys. Part. Nuclei **43**, 882–888 (2012).

[2] M. Piquer i Méndez, "The adiabatic theorem and the Landau–Zener model," B.Sc. thesis, Universitat de Barcelona, Advisor: Josep Taron Roca, 2023.

[3] A. G. Rojo, "Matrix exponential solution of the Landau–Zener problem," arXiv:1004.2914 (2010).

[4] E. P. Glasbrenner and W. P. Schleich, "The Landau–Zener formula made simple," J. Phys. B: At., Mol. Opt. Phys. **56**, 104001 (2023).

[5] Z. K. Silagadze, *Multiple Integral (American Mathematical Monthly Problem 11621 and its Generalization)* (MathOverflow, 2013).

[6] D. Beck, *Interesting Triple Integral* (MathOverflow, 2015).

[7] J. A. d. Reyna, "Cancellations in power series of sine type," arXiv:1505.00440 (2015).